New York University
Washington Square College
Mathematics Research Group

The Relationship between Coaxial and Cylindrical Wave Guide and Cavity Modes

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ABSTRACT

The main result of this paper is the following: Each mode, in a cylindrical guide (or cavity) is approached by a mode in a coaxial guide (or cavity) as the radius of the inner coaxial conductor approaches zero; and conversely, each coaxial mode except the TEM mode approaches a cylindrical mode. The result follows readily from a theorem established in this paper on the behavior of the roots of certain equations involving Bessel functions. Briefly this theorem states that the m^{th} positive root of the equation

$$J_n(x)\, N_n(\rho x) - J_n(\rho x)\, N_n(x) = 0 \qquad (1)$$

approaches the m^{th} positive root of $J_n(x) = 0$ as ρ, the ratio of the radii of the two coaxial conductors, approaches zero. The theorem establishes the same relationship between the roots of

$$J_n'(x)\, N_n'(\rho x) - J_n'(\rho x)\, N_n'(x) = 0 \qquad (2)$$

and those of $J_n'(x) = 0$. Some graphs are included which give actual values of the roots of equations (1) and (2) for varying ρ. These represent extensions of existing tables and should save labor in computing quantities associated with coaxial guides and cavities.

The importance of the theorem established here is twofold. It gives qualitative information on the behavior of coaxial guide and cavity modes. For example, it indicates that the inner conductor of a coaxial guide (or cavity) plays an unimportant part when its radius is small. Secondly, the calculation of cut-off frequencies for coaxial guides, resonant frequencies for coaxial cavities, wave length in a coaxial guide

or cavity, dimensions, etc. may be approximated when the inner conductor is small, by the much simpler computation for the corresponding mode in a cylindrical structure. The theorem also has bearing on the question of standardization of notation for coaxial and cylindrical structures.

of cavity, dielectric, etc. and is satisfactory when the inner conductor is small, by the much simpler computation for the corresponding mode in a cylindrical structure. The Thesis also has bearing on the question of standardization of cavities for coaxial and cylindrical structures.

The Relationship between Coaxial and Cylindrical Wave Guide and Cavity Modes*

1. Introduction. Several conclusions about coaxial modes and their relationship to cylindrical modes have heretofore been based upon very limited physical evidence. In the first place the very existence of modes in coaxial structures depends upon the existence of roots of equations

$$J_n(x)\, N_n(\rho x) - J_n(\rho x)\, N_n(x) = 0, \tag{1}$$

and

$$J_n'(x)\, N_n'(\rho x) - J_n'(\rho x)\, N_n'(x) = 0, \tag{2}$$

for all values of ρ, the ratio of the radii of the conductors, between 0 and 1. No mathematical proof of the existence of such roots has appeared before this time.

Moreover, while engineers have no doubt surmised, and perhaps employed, the fact that each coaxial guide and cavity mode approaches a uniquely corresponding cylindrical mode as the radius of the inner conductor approaches zero, no proof seems to have been given of this fact, nor, indeed has any clear statement been made as to what is meant by one mode approaching another.

This paper will be concerned with the mathematical proof of the existence of coaxial modes for small values of ρ, and with the relationship of these coaxial modes to those which exist in a cylindrical structure, that is, a coaxial structure in which $\rho = 0$. The conclusions rest essentially upon a theorem involving the roots of equations (1) and (2) for small ρ.

*In order to avoid the lengthy and awkward terms "circularly cylindrical coaxial" and "circularly cylindrical", we shall designate these shapes simply "coaxial" and "cylindrical".

This theorem further establishes that such quantities as cut-off frequencies, dimensions, and guide wave lengths for coaxial modes with small ρ are practically those for circular modes.

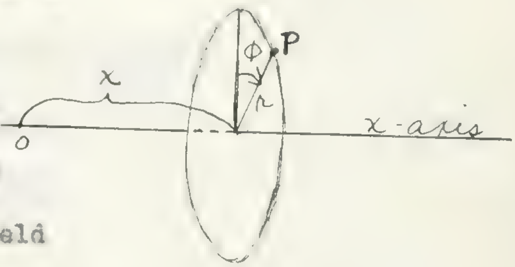
7. The Field Expressions for Cylindrical and Coaxial Modes. For reference in this paper, and elsewhere, the field expressions for guides and cavities of circular coaxial shape, and those of circular cylindrical shape are given here.

The explicit field expressions for cylindrical guides are derived in Stratton, J. A.: Electromagnetic Theory pp. [illegible]4-5 and in Sarbacher, R. I. & William A. Edson: Hyper and Ultra-High Frequency Engineering, pp. 248 & 258. The field expressions for circular cylindrical cavities are derived in Borgnis, F.: Electromagnetic Waves in Dielectric Spaces. Annalen Der Physik, Vol. 35, pp. 359-384, Berlin, 1939.

By using the method in Stratton one obtains the expressions for coaxial guides, and, by Borgnis' method, those for the coaxial cavity are obtained. It is also possible to obtain expressions for the cavity fields by combining guide fields so as to satisfy the changed conditions in the cavity.

The following field expressions presuppose perfect conductors for the walls and a perfect dielectric ($\sigma = 0$) inside. Further, it is understood that the real part of each expression is wanted, the use of $e^{jz} = \cos z + j \sin z$, instead of $\cos z$, being merely a convenience. In the case of the field expressions for cavities it is further understood that the field at any point varies sinusoidally (with a phase determined by the field when $t = 0$). Hence each component should be multiplied by $e^{j\omega t}$ with the understanding again that only the real part of $e^{j\omega t}$ is wanted.

In the field expressions the symbols have the following meanings. In the first place, cylindrical coordinates are used. In this system of coordinates each point in space is represented by the three quantities x, r, ϕ, pictured herewith, where ϕ lies in a plane perpendicular to x - direction and is measured from some arbitrary fixed line in that plane. The field expressions for guides presuppose that the axis of the cylindrical coordinate system lies along the axis of the guide, the zero point on the axis being any desired point or one which meets some given initial condition. The field expressions for cavities presuppose, in addition, that the zero point on the x-axis lies in the plane of one end wall of the cavity.



In this cylindrical coordinate system the field vector $\underline{E}$ is resolved into the components E_x, E_r, and E_ϕ. E_x and E_r are along the x- and r-directions respectively. E_ϕ at any point is perpendicular to E_x and E_r at that point. The same applies to the vector $\underline{H}$.

The special letters used have the following meanings:

A = An amplitude constant whose value depends upon the energy put into the guide or cavity.

J_n = n^{th} order Bessel function of the first kind.

N_n = " " " " " " second " .

J'_n = Derivative of J_n with respect to the argument.

N'_n = " " N_n " " " " " .

a = Radius of the inner coaxial conductor.

b = " " " outer " " (or of the circular cylinder)

n, m, ℓ = Any positive integers (0 is also a permissible value for n and in the case of transverse magnetic modes for ℓ also.)

$j = \sqrt{-1}$

ω = 2π times the frequency of radiation employed.

ε, μ are inductive capacities of the medium.

$s_{n,m}$ = m^{th} positive root of $J_n(x) = 0$

$s'_{n,m}$ = " " " " $J'_n(x) = 0$

$r_{n,m}$ = " " " " equation (1)

$r'_{n,m}$ = " " " " " (2)

$\beta_{n,m} = \sqrt{\omega^2 \varepsilon\mu - \left(\frac{r_{n,m}}{b}\right)^2}$ for COAXIAL STRUCTURES ~~guides~~. Replace $r_{n,m}$ by $s_{n,m}$ for ~~cavities~~. CYLINDRICAL STRUCTURES

$\beta'_{n,m} = \sqrt{\omega^2 \varepsilon\mu - \left(\frac{r'_{n,m}}{b}\right)^2}$ for COAXIAL STRUCTURES ~~guides~~. Replace $r'_{n,m}$ by $s'_{n,m}$ for ~~cavities~~. CYLINDRICAL STRUCTURES.

x_e = length of cavity.

Field Expressions for Cylindrical Guides

Transverse Magnetic Modes

$$H_x \equiv 0$$

$$E_x = A \cos n\phi \, J_n\left(r \frac{s_{n,m}}{b}\right) e^{j(wt - \beta_{n,m} x)}$$

$$E_r = -jA \, \beta_{n,m} \left(\frac{b}{s_{n,m}}\right) \cos n\phi \, J_n'\left(r \frac{s_{n,m}}{b}\right) e^{j(wt - \beta_{n,m} x)}$$

$$E_{\phi} = jA \, \beta_{n,m} \left(\frac{b}{s_{n,m}}\right)^2 \frac{n}{r} \sin n\phi \, J_n\left(r \frac{s_{n,m}}{b}\right) e^{j(wt - \beta_{n,m} x)}$$

$$H_r = -jA \, \omega\varepsilon \left(\frac{b}{s_{n,m}}\right)^2 \frac{n}{r} \sin n\phi \, J_n\left(r \frac{s_{n,m}}{b}\right) e^{j(wt - \beta_{n,m} x)}$$

$$H_{\phi} = -jA \, \omega\varepsilon \left(\frac{b}{s_{n,m}}\right) \cos n\phi \, J_n'\left(r \frac{s_{n,m}}{b}\right) e^{j(wt - \beta_{n,m} x)}$$

Transverse Electric Modes

$$E_x \equiv 0$$

$$E_r = jA\omega\mu \left(\frac{b}{s'_{n,m}}\right)^2 \frac{n}{r} \sin n\phi \, J_n\left(r \frac{s'_{n,m}}{b}\right) e^{j(wt - \beta'_{n,m} x)}$$

$$E = jA\omega\mu \left(\frac{b}{s'_{n,m}}\right) \cos n\phi \, J_n'\left(r \frac{s'_{n,m}}{b}\right) e^{j(wt - \beta'_{n,m} x)}$$

$$H_x = A \cos n\phi \, J_n\left(r \frac{s'_{n,m}}{b}\right) e^{j(wt - \beta'_{n,m} x)}$$

$$H_r = -jA \, \beta'_{n,m} \left(\frac{b}{s'_{n,m}}\right) \cos n\phi \, J_n'\left(r \frac{s'_{n,m}}{b}\right) e^{j(wt - \beta'_{n,m} x)}$$

$$H_{\phi} = jA \, \beta'_{n,m} \left(\frac{b}{s'_{n,m}}\right)^2 \frac{n}{r} \sin n\phi \, J_n\left(r \frac{s'_{n,m}}{b}\right) e^{j(wt - \beta'_{n,m} x)}$$

Field Expressions for Coaxial Guides

Transverse Magnetic Modes

$$H_x \equiv 0$$

$$E_x = A \cos n\phi \left[J_n\left(r \frac{r_{n,m}}{b}\right) - \alpha N_n\left(r \frac{r_{n,m}}{b}\right) \right] e^{j(wt-\beta_{n,m}x)}$$

$$E_r = -jA\beta_{n,m} \left(\frac{b}{r_{n,m}}\right) \cos n\phi \left[J'_n\left(r \frac{r_{n,m}}{b}\right) - \alpha N'_n\left(r \frac{r_{n,m}}{b}\right) \right] e^{j(wt-\beta_{n,m}x)}$$

$$E_\phi = jA\beta_{n,m} \left(\frac{b}{r_{n,m}}\right)^2 \frac{n}{r} \sin n\phi \left[J_n\left(r \frac{r_{n,m}}{b}\right) - \alpha N_n\left(r \frac{r_{n,m}}{b}\right) \right] e^{j(wt-\beta_{n,m}x)}$$

$$H_r = -jA\omega\varepsilon \left(\frac{b}{r_{n,m}}\right)^2 \frac{n}{r} \sin n\phi \left[J_n\left(r \frac{r_{n,m}}{b}\right) - \alpha N_n\left(r \frac{r_{n,m}}{b}\right) \right] e^{j(\omega t-\beta_{n,m}x)}$$

$$H_\phi = -jA\omega\varepsilon \left(\frac{b}{r_{n,m}}\right) \cos n\phi \left[J'_n\left(r \frac{r_{n,m}}{b}\right) - \alpha N'_n\left(r \frac{r_{n,m}}{b}\right) \right] e^{j(wt-\beta_{n,m}x)}$$

$$\alpha = \frac{J_n(r_{n,m})}{N_n(r_{n,m})} = \frac{J_n\left(\frac{a}{b} r_{n,m}\right)}{N_n\left(\frac{a}{b} r_{n,m}\right)}$$

Tranverse Electric Modes

$$E_x \equiv 0$$

$$E_r = jA\omega\mu \left(\frac{b}{r'_{n,m}}\right)^2 \frac{n}{r} \sin n\phi \left[J_n\left(r \frac{r'_{n,m}}{b}\right) - \alpha N_n\left(r \frac{r'_{n,m}}{b}\right) \right] e^{j(wt-\beta'_{n,m}x)}$$

$$H_x = A \cos n\phi \left[J_n\left(r \frac{r'_{n,m}}{b}\right) - \alpha N_n\left(r \frac{r'_{n,m}}{b}\right) \right] e^{j(wt-\beta'_{n,m}x)}$$

$$E_\phi = jA\omega\mu \left(\frac{b}{r'_{n,m}}\right) \cos n\phi \left[J'_n\left(r \frac{r'_{n,m}}{b}\right) - \alpha N'_n\left(r \frac{r'_{n,m}}{b}\right) \right] e^{j(wt-\beta'_{n,m}x)}$$

$$H_r = -jA\beta'_{n,m} \frac{b}{r'_{n,m}} \cos n\phi \left[J'_n\left(r \frac{r'_{n,m}}{b}\right) - \alpha N'_n\left(r \frac{r'_{n,m}}{b}\right) \right] e^{j(wt-\beta'_{n,m}x)}$$

$$H_\phi = jA\beta'_{n,m} \left(\frac{b}{r'_{n,m}}\right)^2 \frac{n}{r} \sin n\phi \left[J_n\left(r \frac{r'_{n,m}}{b}\right) - \alpha N_n\left(r \frac{r'_{n,m}}{b}\right) \right] e^{j(wt-\beta'_{n,m}x)}$$

$$\alpha = \frac{J'_n(r'_{n,m})}{N'_n(r'_{n,m})} = \frac{J'_n\left(\frac{a}{b} r'_{n,m}\right)}{N'_n\left(\frac{a}{b} r'_{n,m}\right)}$$

Field Expressions for Cylindrical Cavities

Transverse Magnetic Modes

$$H_x \equiv 0$$

$$E_x = A \left(\frac{s_{n,m}}{b}\right)^2 \cos n\phi \, J_n\!\left(r \frac{s_{n,m}}{b}\right) \cos \frac{l\pi x}{x_0}$$

$$E_r = -A \frac{n\pi}{x_0} \frac{s_{n,m}}{b} \cos n\phi \, J_n'\!\left(r \frac{s_{n,m}}{b}\right) \sin \frac{l\pi x}{x_0}$$

$$E_\phi = A \frac{l\pi}{x_0} \frac{n}{r} \sin n\phi \, J_n\!\left(r \frac{s_{n,m}}{b}\right) \sin \frac{l\pi x}{x_0}$$

$$H_r = -A \sqrt{\left(\frac{s_{n,m}}{b}\right)^2 + \frac{l^2\pi^2}{x_0^2}} \; \frac{n}{r} \sin n\phi \, J_n\!\left(r \frac{s_{n,m}}{b}\right) \cos \frac{l\pi x}{x_0}$$

$$H_\phi = -A \frac{s_{n,m}}{b} \sqrt{\left(\frac{s_{n,m}}{b}\right)^2 + \frac{l^2\pi^2}{x_0^2}} \; \cos n\phi \, J_n'\!\left(r \frac{s_{n,m}}{b}\right) \cos \frac{l\pi x}{x_0}$$

Transverse Electric Modes

$$E_x \equiv 0$$

$$E_r = -A \sqrt{\frac{s'_{n,m}}{b}^2 + \frac{l^2\pi^2}{x_0^2}} \; \frac{n}{r} \sin n\phi \, J_n\!\left(r \frac{s'_{n,m}}{b}\right) \sin \frac{l\pi x}{x_0}$$

$$E_\phi = -A \frac{s'_{n,m}}{b} \sqrt{\left(\frac{s'_{n,m}}{b}\right)^2 + \frac{l^2\pi^2}{x_0^2}} \; \cos n\phi \, J_n'\!\left(r \frac{s'_{n,m}}{b}\right) \sin \frac{l\pi x}{x_0}$$

$$H_x = A \left(\frac{s'_{n,m}}{b}\right)^2 \cos n\phi \, J_n\!\left(r \frac{s'_{n,m}}{b}\right) \sin \frac{l\pi x}{x_0}$$

$$H_r = A \frac{l\pi}{x_0} \frac{s'_{n,m}}{b} \cos n\phi \, J_n'\!\left(r \frac{s'_{n,m}}{b}\right) \cos \frac{l\pi x}{x_0}$$

$$H_\phi = -A \frac{l\pi}{x_0} \frac{n}{r} \sin n\phi \, J_n\!\left(r \frac{s'_{n,m}}{b}\right) \cos \frac{l\pi x}{x_0}$$

Field Expressions for Coaxial Cavities

Transverse Magnetic Modes

$$H_x \equiv 0$$

$$E_x = -A\left(\frac{r_{n,m}}{b}\right)^2 \cos n\phi \left[J_n\left(r\frac{r_{n,m}}{b}\right) - \alpha N_n\left(r\frac{r_{n,m}}{b}\right)\right] \cos \frac{\ell\pi x}{x_o}$$

$$E_r = -A\,\frac{n\pi}{x_o}\,\frac{r_{n,m}}{b}\,\cos n\phi \left[J'_n\left(r\frac{r_{n,m}}{b}\right) - \alpha N'_n\left(r\frac{r_{n,m}}{b}\right)\right] \sin \frac{\ell\pi x}{x_o}$$

$$E_\phi = A\,\frac{\ell\pi}{x_o}\,\frac{n}{r}\,\sin n\phi \left[J_n\left(r\frac{r_{n,m}}{b}\right) - \alpha N_n\left(r\frac{r_{n,m}}{b}\right)\right] \sin \frac{\ell\pi x}{x_o}$$

$$H_r = -A\sqrt{\left(\frac{r_{n,m}}{b}\right)^2 + \frac{\ell^2\pi^2}{x_o^2}}\;\frac{n}{r}\,\sin n\phi \left[J_n\left(r\frac{r_{n,m}}{b}\right) - \alpha N_n\left(r\frac{r_{n,m}}{b}\right)\right] \cos \frac{\ell\pi x}{x_o}$$

$$H_\phi = -A\,\frac{r_{n,m}}{b}\sqrt{\left(\frac{r_{n,m}}{b}\right)^2 + \frac{\ell^2\pi^2}{x_o^2}}\;\cos n\phi \left[J'_n\left(r\frac{r_{n,m}}{b}\right) - \alpha N'_n\left(r\frac{r_{n,m}}{b}\right)\right] \cos \frac{\ell\pi x}{x_o}$$

$$\alpha = \frac{J_n(r_{n,m})}{N_n(r_{n,m})} = \frac{J_n\left(\frac{a}{b}\,r_{n,m}\right)}{N_n\left(\frac{a}{b}\,r_{n,m}\right)}$$

Transverse Electric Modes

$$E_x \equiv 0$$

$$E_r = -A\sqrt{\left(\frac{r'_{n,m}}{b}\right)^2 + \frac{\ell^2\pi^2}{x_o^2}}\;\frac{n}{r}\,\sin n\phi \left[J_n\left(r\frac{r'_{n,m}}{b}\right) - \alpha N_n\left(r\frac{r'_{n,m}}{b}\right)\right] \sin \frac{\ell\pi x}{x_o}$$

$$E_\phi = -A\,\frac{r'_{n,m}}{b}\sqrt{\left(\frac{r'_{n,m}}{b}\right)^2 + \frac{\ell^2\pi^2}{x_o^2}}\;\cos n\phi \left[J'_n\left(r\frac{r'_{n,m}}{b}\right) - \alpha N'_n\left(r\frac{r'_{n,m}}{b}\right)\right] \sin \frac{\ell\pi x}{x_o}$$

$$H_x = A\,\frac{r'_{n,m}}{b}\,\cos n\phi \left[J_n\left(r\frac{r'_{n,m}}{b}\right) - \alpha N_n\left(r\frac{r'_{n,m}}{b}\right)\right] \sin \frac{\ell\pi x}{x_o}$$

$$H_r = A\,\frac{\ell\pi}{x_o}\,\frac{r'_{n,m}}{b}\,\cos n\phi \left[J'_n\left(r\frac{r'_{n,m}}{b}\right) - \alpha N'_n\left(r\frac{r'_{n,m}}{b}\right)\right] \cos \frac{\ell\pi x}{x_o}$$

$$H_\phi = -A\,\frac{\ell\pi}{x_o}\,\frac{n}{r}\,\sin n\phi \left[J_n\left(r\frac{r'_{n,m}}{b}\right) - \alpha N_n\left(r\frac{r'_{n,m}}{b}\right)\right] \cos \frac{\ell\pi x}{x_o}$$

$$\alpha = \frac{J'_n(r'_{n,m})}{N'_n(r'_{n,m})} \quad \frac{J'_n\left(\frac{a}{b}\,r'_{n,m}\right)}{N'_n\left(\frac{a}{b}\,r'_{n,m}\right)}$$

Field Expressions for the TEM Coaxial Guide Mode

$$H_x \equiv 0$$

$$E_x \equiv 0$$

$$H_\phi = \frac{A}{r} \, e^{j(\omega t - \beta x)}$$

$$E_\phi \equiv 0$$

$$E_r = \frac{A}{2\pi r} \sqrt{\frac{\mu}{\varepsilon}} \, e^{j(\omega t - \beta x)}$$

$$H_r \equiv 0$$

$\beta = \dfrac{2\pi}{\lambda}$ where λ is the free space wave length in the dielectric of the guide.

Field Expressions for the TEM Coaxial Cavity Modes

$$H_x \equiv 0$$

$$E_x \equiv 0$$

$$H_\phi = \frac{A}{r} \cos \frac{\ell \pi x}{x_o} \qquad \ell = 1, 2, 3, \ldots\ldots$$

$$E_r = \frac{A}{2\pi r} \sqrt{\frac{\mu}{\varepsilon}} \sin \frac{\ell \pi x}{x_o}$$

$$H_r \equiv 0$$

$$E_\phi \equiv 0$$

As the field expressions suggest each guide mode, Transverse Magnetic or Transverse Electric, may be designated by two indices, the first referring to the order of the Bessel functions involved in its expressions and the second to the order of the root of the appropriate Bessel function equation. Solutions of equation (1) of article 1 correspond to Transverse Magnetic modes and are denoted by $r_{n,m}$, while solutions of equation (2) correspond to Transverse Electric modes and are denoted by $r'_{n,m}$. In the case of cavity modes the first two indices have the same meaning and the third is the value given to ℓ in the field expressions. In accordance with these conventions, the indices assigned to a mode depend entirely upon the mathematical expressions for the field and not at all on any physical meanings which may, in addition, be attached to them.

3. The Roots of the Transcendental Equation. It is seen from the field expressions that the quantities $r_{n,m}$, ~~and~~ $r'_{n,m}$, $s_{n,m}$, and $s'_{n,m}$ are involved essentially in the mathematical representation of guide and cavity modes. Moreover, since $r_{n,m}$ and $r'_{n,m}$ depend upon the ratio of the radii of the coaxial guides and cavities, we must know how these r's vary if we are to study the variation of the field expressions with the radii.

The values of $r_{n,m}$ for consecutive integral values of m are the consecutive values of x satisfying equation (1), while the values of $r'_{n,m}$ are the values of x satisfying equation (2). Such roots, if they exist, of course depend upon the value of ρ. Now the existence of a coaxial mode, for various values of ρ, depends upon the existence of a solution $x = x(\rho)$, of equation (1) [or (2)]. While it is logically possible for x to vary quite irregularly as ρ varies, the only variation

which would be of physical interest or likely to occur on physical grounds* is a continuous variation of x with ρ such that x approaches some definite, non-zero limit as ρ approaches 0. [The limit of x cannot be 0 because the field expressions for coaxial guides show that in that case the amplitude of the field components would have to become infinite as ρ approaches zero. The limit cannot be infinite either, for in that event, the field expressions for coaxial cavities show that the field components would become infinite.] Considering, then, only such roots of equations (1) and (2) as vary continuously with ρ and approach finite, non-zero limits as ρ tends to zero, we state (1) that they exist, and (2) that the limits approached are the roots of $J_n(x) = 0$ and $J_n'(x) = 0$. This is made precise by the following theorem, proof of which is given in the appendix:

<u>Theorem:</u> The m^{th} positive root (in order of magnitude) of

$$J_n(x)\, N_n(\rho x) - J_n(\rho x)\, N_n(x) = 0 \qquad (1)$$

exists for all ρ sufficiently small and approaches the m^{th} positive root of $J_n(x) = 0$ as ρ approaches 0. The same statement relates the roots of

$$J_n'(x)\, N_n'(\rho x) - J_n'(\rho x)\, N_n'(x) = 0 \qquad (2)$$

and those of $J_n'(x)=0$.

4. <u>The Relationship between Coaxial and Cylindrical Modes</u>. The proof that each coaxial mode (except the TEM or principal mode) approaches a corresponding cylindrical mode, and that each cylindrical mode is approached by a coaxial one, will now be made by appealing directly to the field expressions. The sense in which this approach is to be

* Graphical analysis of equations (1) and (2) also confirms this point.

taken is as follows. Consider any point in a coaxial guide that is, fix x, r, and ϕ. Then as the radius of the inner conductor tends to 0, each field component, E_x, E_ϕ, etc., at that point approaches the corresponding field component of the cylindrical mode with the same indices.

Consider as an illustration the expression for E_x in a transverse magnetic mode of a coaxial guide:

$$E_x = A \cos n\phi \left[J_n(r \frac{r_{n,m}}{b}) - \alpha N_n(r \frac{r_{n,m}}{b}) \right] e^{j(wt - \beta_{n,m} x)},$$

where $\alpha = + \dfrac{J_n(\frac{a}{b} r_{n,m})}{N_n(\frac{a}{b} r_{n,m})}$.

$r_{n,m}$, it will be remembered, is the m^{th} root of

$$J_n(x) N_n(\rho x) - J_n(\rho x) N_n(x) = 0,$$

where $\rho = \frac{a}{b}$ and, of course, varies with ρ.

It has been proved that $r_{n,m}$ exists for small ρ and that, as ρ approaches zero, $r_{n,m}$ approaches the m^{th} root of $J_n(x) = 0$. Hence, by reference to the expression for α, we see that as $\frac{a}{b}$ tends to 0 the argument in the numerator tends to 0. Moreover, $J_n(x)$ approaches 0 as x tends to 0 for $n \geq 1$; while $J_o(x)$ approaches 1 as x tends to 0. Hence the numerator in the expression for α approaches 0 or 1. The argument in the denominator likewise approaches 0, and $N_n(x)$ tends to ∞ as x tends to 0. Hence, α tends to 0. Moreover, $J_n(r \frac{r_{n,m}}{b})$ and $N_n(r \frac{r_{n,m}}{b})$ remain BOUNDED ~~finite~~ as $r_{n,m}$ varies while r is fixed. It should be noted that $\beta_{n,m}$ varies with

$\frac{a}{b}$ because $r_{n,m}$ does. However, $r_{n,m}$ for a coaxial guide approaches $s_{n,m}$ of the cylindrical guide by the theorem in the preceding section. The behavior of $r_{n,m}$ is pertinent also where it appears explicitly in the field expressions or as part of the argument of some function. Hence, the field expression for E_x becomes precisely the expression for E_x in the cylindrical guide mode designated by the same n and m values.

The argument of the preceding paragraph applies without exception to each field component of each guide and cavity mode, transverse electric as well as transverse magnetic, for the theorem on the behavior of $r_{n,m}$ applies to $r'_{n,m}$ also.

The converse conclusion that to each cylindrical mode there corresponds a coaxial mode which approaches the cylindrical one follows from the fact that to each $s_{n,m}$ (or $s'_{n,m}$) there is an $r_{n,m}$ (or $r'_{n,m}$) which then determines some coaxial mode. Then, as above, the field expressions of that coaxial mode must approach those of the cylindrical mode.

The conclusion reached in this article is somewhat contrary to intuition. If the coaxial conductors have perfect conductivity no mode can exist in which the electric field has a component parallel to the inner (or outer) conductor AT THE CONDUCTOR. This state of affairs must hold no matter how thin the inner conductor is. Nevertheless some coaxial modes approach circular modes which do possess a longitudinal component of the electric field ALONG THE AXIS.

This surprising fact may be made more acceptable to the intuition if one notes the expression for E_x above. For r = a, E_x is of course 0, as the boundary condition requires. However, for r not equal to a, the

term N_n subtracts from J_n to determine E_x. But for $\frac{a}{b}$ near 0 the value of α is very small since $N_n(\frac{a}{b}\ r_{n,m})$ tends to ∞ as $\frac{a}{b}$ approaches 0. Hence if r is kept fixed and $\frac{a}{b}$ is allowed to approach 0, the term $\alpha N_n(\frac{r\,r_{n,m}}{b})$ subtracts less and less from J_n, for J_n and N_n change little while α approaches 0 very rapidly with $\frac{a}{b}$.

This discussion implies that at any fixed position in a coaxial guide the thinner the inner conductor the less significant is its effect at that fixed position. Hence, the electric field at that position may well tend to have a larger and larger longitudinal component as $\frac{a}{b}$ approaches 0. This argument may be applied at any fixed point in the guide as near to the inner conductor as one pleases. Hence, one sees how a longitudinal component of $\underline{E}$ may well establish itself as the radius of the inner conductor approaches zero.

One may state the fact of the previous paragraph in a somewhat more intuitive form, namely, that the insertion of an inner conductor in a cylindrical guide (or cavity) has very little effect on the field except in its own immediate neighborhood.

From the theorem stated in article 3 on the behavior of the roots $r_{n,m}$ and $r'_{n,m}$ we have just drawn the useful conclusion that the field in a coaxial guide (or cavity) with a thin inner conductor is almost the field in a cylindrical guide (or cavity). The theorem is useful also in giving us information on cut-off frequencies, wave length in the guide, dimensions, and propagation constants. For example, the cut-off frequencies for the $TM_{n,m}$ and $TE_{n,m}$ modes in a coaxial guide are given, respectively, by

$$f_c = \frac{c\,r_{n,m}}{2\pi b}, \quad \text{and} \quad f_c = \frac{c\,r'_{n,m}}{2\pi b}, \tag{1}$$

where c is the velocity of light in the dielectric of the guide, b is the radius of the outer conductor, and $r_{n,m}$ and $r'_{n,m}$ are the roots discussed in article 3. For cylindrical guides the cut-off frequencies are given by

$$f_c = \frac{c\,s_{n,m}}{2\pi b} \quad \text{and} \quad f_c = \frac{c\,s'_{n,m}}{2\pi b}$$

where b is the radius of the cylinder.

Since $r_{n,m}$ approaches $s_{n,m}$ and $r'_{n,m}$ approaches $s'_{n,m}$ as the radius of the inner coaxial conductor approaches zero, we see that the cut-off frequency of a coaxial guide mode approaches the cut-off frequency of the associated cylindrical mode as the inner conductor shrinks in diameter. Of course these formulas may be used to determine the critical dimensions in terms of a given frequency, and again the limiting relationship applies.

The expressions for the cut-off wave-lengths and propagation constants in a coaxial guide and the expressions for the resonant frequencies in coaxial cavities likewise depend upon $r_{n,m}$ and $r'_{n,m}$, and these expressions approach the corresponding expressions for cylindrical guides and cavities because, as above, $r_{n,m}$ approaches $s_{n,m}$ and $r'_{n,m}$ approaches $s'_{n,m}$.

5. Graphical Representation of $r_{n,m}$ and $r'_{n,m}$.

The theorem of article 3 establishes many useful conclusions but does not in itself give any precise quantitative information for the calculation of such quantities as dimensions or frequencies required in guide and cavity work. Since the calculation of $r_{n,m}$ and $r'_{n,m}$ for any given value of ρ is quite laborious and very few of these values have been recorded* it was decided to make some quantitative record of the behavior of r and r' in terms of ρ, for at least the lowest modes. The graphs which are attached show these relationships for n = 0,1,2,3 and m = 1,2. It should be noted that the graphs showing the relationship between $r'_{0,m}$ and ρ are the same as those showing the relationship between $r_{1,m}$ and ρ.

* A few $r_{n,m}$ values may be found in Jahnke-Emde: Tables of Functions, pp. 204-6. A few graphs have also appeared in the literature but only for specific, limited purposes.

By using these graphs and the appropriate formulas one may obtain fairly accurately the dimensions or resonant frequencies which one would have to employ for a given coaxial structure. However, to compute dimensions to the nearest thousandth of an inch, for example, the data must be refined. The graphs submitted here should be used as points of departure for further calculation if very precise numerical values are wanted. For higher modes, lack of sufficiently extensive Bessel function tables does not permit even fairly good calculations at present.

For large $r_{n,m}$ and $r'_{n,m}$ values there are asymptotic expressions developed by McMahon*. Our attempts to improve these expressions so that they might be usable for small r and r' have thus far been unsuccessful. Such improved expressions would be desirable for they would permit substitution of the value of ρ and a fairly quick calculation of r or r'. Moreover, their importance extends beyond the application to guide and cavity theory, for these roots are useful in acoustical and hydrodynamical problems also.

One incidental point connected with carrying out the calculation of the graphs may be worth noting. Since Bessel function tables are not very extensive the following device can be used. Equation (2) of article 3 can be written thus:

$$J'_n(x) = N'_n(x) \cdot \frac{J'_n(\rho x)}{N'_n(\rho x)} . \qquad (2)$$

Let s_p and s_{p+1} be two successive roots of $J'_n(x) = 0$. When $\rho = \dfrac{s_p}{s_{p+1}}$ and $x = s_{p+1}$, equation (2) is satisfied. Hence, for this value of ρ,

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* McMahon, J.: "On the Roots of the Bessel and Certain Related Functions", Annals of Mathematics, Vol. 9, 1894-95, pp. 23-30.

s_{p+1} is a root of (2) as well as of $J'_n(x)$. By letting p take on the values 1, 2, ... the calculation of the second root** of equation (2) for any ρ values falls back on a knowledge of the roots of $J'_n(x)$ itself. By using the ratio $\dfrac{s_p}{s_{p+2}}$ and letting p take on the values of 1,2,3,.... we obtain the third root of (1) in terms of ρ; Etc.

The process used above has the following possible physical application. For the particular ρ values $\dfrac{s_p}{s_{p+1}}$, say, used above, the corresponding second roots of equation (1) are also roots of $J'_n(x)$. Hence for these ρ values the coaxial structure has the same frequency (cut-off or resonant), propagation constant, and internal wave length, though not the same field expressions, as some cylindrical mode in a cylinder whose radius is the outer radius of the coaxial pair. One may therefore convert a cylindrical guide (or cavity) into a coaxial one having the same cut-off frequency (or resonant frequency), internal wave length, propagation constant, etc., by choosing the ratio of the radii of the coaxial conductors properly.

A rather significant conclusion, not readily predictable theoretically, is evident from these graphs in conjunction with formula (1) of article 4. In the case of coaxial $TE_{n,1}$ modes the cut-off frequencies decrease as $\frac{a}{b}$ increases. That is, lower wave-lengths may be accommodated. However, for the coaxial transverse magnetic modes as well as the other transverse electric modes, increasing $\frac{a}{b}$ increases the cut-off frequency to infinity. The same remark applies to the resonant frequencies of cavities.

**The fact that it is the second root follows from graphical considerations.

6. Further Significance of the Relation between Cylindrical and Coaxial Modes.

The theorem establishing that each coaxial mode approaches a cylindrical mode as the radius of the inner conductor approaches zero makes the convention for designating the modes that was adopted in article 2 of this report, seem quite reasonable. Coaxial and cylindrical modes which are associated by the theorem are designated by the same indices. That is the $TE_{3,2,1}$ coaxial cavity mode, for example, approaches the $TE_{3,2,1}$ cylindrical cavity mode as the radius of the inner coaxial conductor approaches zero. This point of view has also been urged by Mr. W. H. Huggins of the Watson Laboratories' Cambridge Field Station, who has gone further to show how the indices so chosen may be interpreted physically for coaxial and circular guides and cavities. Mr. Huggins' suggestions replace incorrect ones made by Barrow and Wieher in 1940* and since then incorporated in texts.**

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* Barrow, W.L., and W.H. Mieher: "Natural Oscillations of Electrical Cavity Resonators," Proc. I.R.E., Vol. 28, 1940, p. 184.

** Sarbacher, R.I., and W.A. Edson: Hyper and Ultrahigh Frequency Engineering, p. 387

APPENDIX

We have to prove the following theorem, stated in article 3:

__Theorem.__ The m^{th} positive root of

$$J_n(x)\, N_n(\rho x) - J_n(\rho x)\, N_n(x) = 0 \qquad (1)$$

exists for all ρ sufficiently small, and approaches the m^{th} positive root of $J_n(x) = 0$ as ρ approaches 0. The same statement relates the roots of

$$J_n'(x)\, N_n'(\rho x) - J_n'(\rho x)\, N_n'(x) = 0 \qquad (2)$$

and those of $J_n'(x) = 0$.

__Proof.__ The proof will be divided into two parts. First, to each root $s_{n,m}$ of $J_n(x) = 0$ there will be shown to be one and only one continuous function $x(\rho)$ which is a solution of equation (1) and such that, as ρ approaches 0, x approaches $s_{n,m}$. The same will then be shown for $J_n'(x)$ and equation (2). Of course these $x(\rho)$ are the $r_{n,m}$ and $r_{n,m}'$ which very with ρ.

We then show that to each root of equation (1) [or (2)], in the sense explained in article 3 there exists one and only one root of $J_n(x) = 0$ [or $J_n'(x) = 0$]. Since the roots of $J_n(x) = 0$ [and of $J_n'(x) = 0$] may be ordered with respect to magnitude, and these are in one-to-one correspondence with the roots of equations (1) [or (2)], we may say that the m^{th} root of $J_n(x) = 0$ is related by the theorem to the m^{th} root of equation (1), and the m^{th} root of $J_n'(x) = 0$ is related in the same way to the m^{th} root of equation (2).

The first part of the theorem depends upon an implicit function theorem which states that if $F(x,y)$ is a continuous function of x and y in the neighborhood of a point (x_0, y_0), if $F(x_0, y_0) = 0$, and if $F_y(x_0, y_0)$ is continuous and different from zero at (x_0, y_0), then there exists a unique function $y = f(x)$ defined and continuous in a neighborhood of x_0 and such that $F\left[x, f(x)\right] \equiv 0$ in that neighborhood.*

*A proof of this theorem may be found in Courant, R.: __Differential and Integral Calculus__, Vol. II, pp. 119-121.

We shall apply this theorem to establish the existence of a root of equation (1) above which approaches a given root of $J_n(x) = 0$. The point (x_0, y_0) of the implicit function theorem will be $\rho = 0$ and $x = s_{n,m}$ where $s_{n,m}$ is the m^{th} positive root of $J_n(x) = 0$ in order of magnitude. We shall, however, apply the theorem to a neighborhood of $(0, s_{n,m})$ which extends from 0 only on the positive side of 0, as far as ρ values are concerned. This use of the implicit function theorem is quite common in advanced analysis.

For convenient reference we write the explicit expression for $N_n(z)$:

$$N_n(z) = - \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} \left(\frac{z}{2} \right)^{-n+2m}$$

$$+ \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\left(\frac{z}{2}\right)^{n+2m}}{m!(n+m)!} \left\{ 2 \log \frac{z}{2} - \psi(m+1) - \psi(n+m+1) \right\}$$

$$N_0(z) = \left[\gamma + \log \frac{z}{2} \right] J_0(z) - \sum_{m=1}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{2m}}{(m!)^2} \left\{ 1 + \frac{1}{2} + \dots + \frac{1}{m} \right\}$$

In these expressions $\psi(m+1)$ and $\psi(n+m+1)$ are independent of z, and is just a constant.

It will be noticed that at $z = 0$, $N_n(z)$ has a discontinuity arising from two kinds of terms, those involving negative powers of z up to z^{-n} and those having log z as a multiplier. This singularity interferes with the application of the implicit function theorem. This difficulty can be avoided by the insertion of a factor which has no effect on the conclusion.

Hence, we consider the following expression:

$$\rho^n \left[J_n(x) N_n(\rho x) - J_n(\rho x) N_n(x) \right] = 0, n = 1, 2, 3, \dots \quad (3)$$

The case of n=0 will be considered separately. This new expression is defined as follows for $\rho = 0$. The factor ρ^n is to be associated with $N_n(\rho x)$ and $N_n(x)$,

and the value of $\rho^n N_n(\rho x)$ for $\rho = 0$ is defined as $\lim_{\rho \to 0} \rho^n N_n(\rho x)$. Since $\lim_{\rho \to 0} \rho^n \log \rho$ is 0, and multiplication by ρ^n removes all negative powers of ρ in $\rho^n N_n(\rho x)$, this function becomes continuous for all $\rho \geq 0$ and for all positive values of x.

We note also that a solution of equation (3), for x in terms of ρ, must also satisfy equation (1) for $\rho \neq 0$. Hence if we establish the existence of the desired root of equation (3) which approaches a given $s_{n,m}$ we shall have established the conclusion for equation (1) also.

Let us call the left side of equation (3), $f(\rho, x)$. Then we note that $f(\rho, x)$ is continuous for all $\rho \geq 0$ and for all positive x. Also, $f(0, s_{n,m}) = 0$.

We now prove that $\frac{\partial f}{\partial x}(0, s_{n,m}) \neq 0$. We shall calculate this derivative by finding $\frac{\partial f}{\partial x}$ at $(\rho, s_{n,m})$, which is obtainable by straightforward differentiation for $\rho \neq 0$, and then finding $\lim_{\rho \to 0} \frac{\partial f}{\partial x}(\rho, s_{n,m})$. This limit is taken as the value of the function $\frac{\partial f(\rho, x)}{\partial x}$ at $(0, s_{n,m})$, and this limit is $\frac{\partial f}{\partial x}(0, s_{n,m})$, as shown by the lemma below.

For $\rho \neq 0$, then,

$$\frac{\partial f}{\partial x}(\rho, x) = \rho^n \left[J_n'(x) N_n(\rho x) + J_n(x) \frac{\partial}{\partial x} N_n(\rho x) - \frac{\partial}{\partial x} J_n(\rho x) N_n(x) - J_n(\rho x) \frac{\partial}{\partial x} N_n(x) \right]$$

Examination of the terms of this function show that the function is continuous at $\rho = 0$, $x = s_{n,m}$. If we now consider the limit of each of these terms as ρ approaches zero and remember that $J_n(s_{n,m}) = 0$, we find that all but the term $\rho^n J_n'(x) N_n(\rho x)$ are zero. The value of this term in view of the expression for $N_n(z)$ is $J_n'(s_{n,m}) \left[-(n-1)! \left(\frac{2}{s_{n,m}} \right)^n \right]$ Now it is well known*

*This follows immediately from the relation, $J_n'(x) = \frac{n}{Z} J_n(Z) - J_{n+1}(Z)$ together with the fact that the zeros of $J_n(z)$ and $J_{n+1}(z)$ separate each other.

that $J_n'(x)$ cannot be 0 for those values of x for which $J_n(x) = 0$. Hence, this term is not zero and in view of the lemma already referred to, $\frac{\partial f}{\partial x}(0, s_{n,m}) \neq 0$.

By the implicit function theorem we may assert the existence of a unique continuous function $x(\rho)$ such that $f\left[\rho, x(\rho)\right] \equiv 0$, $x(0) = s_{n,m}$, and $\lim_{\rho \to 0} x(\rho) = s_{n,m}$. Hence, as remarked above, we have the existence of a root of (1) which, of course, varies with ρ, and which approaches a root of $J_n(x) = 0$.

The case of equation (1) for $n = 0$ may be handled in exactly the same manner except that equation (3) is replace by

$$\frac{1}{\log \rho}\left[J_o(x)\, N_o(\rho x) - J_o(\rho x)\, N_o(x)\right] = 0 . \qquad (4)$$

By differentiating this expression with respect to x, taking the limit as ρ tends to 0 and using the facts that $J_o(0) = 1$ and $J_o'(s_{o,m}) \neq 0$, we again obtain the hypotheses necessary to assert the existence of a solution, $x(\rho)$ of equation (4) associated with $s_{o,m}$ and approaching $s_{o,m}$ as ρ approaches 0.

We may now consider the case of equation (2). First let us note the value of $N_n'(z)$ for $n = 1,2,3,\ldots\ldots$.

$$N_n'(z) = -\sum_{m=0}^{n-1} \frac{(-n+2m)(n-m-1)\,!}{m!}\left(\frac{z}{2}\right)^{-n+2m-1}$$

$$+ \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\left(\frac{z}{2}\right)^{n+2m-1}}{m!\,(n+m)!}\left[2 \log \frac{z}{2} + 1 - \psi(m+1) - \psi(n+m+1)\right].$$

We see that the case of $N_n'(z)$ is exactly like ~~that~~ of $N_n(z)$ except that the power of z which, together with log z, causes a discontinuity as $z = 0$, is z^{-n-1} instead of z^{-n}. Hence, we replace ρ^n in equation (3) by ρ^{n+1} and proceed as before. In place of $s_{n,m}$ we use $s_{n,m}'$, the zeros of $J_n'(x)$. We shall

need this time the fact that $J_n''(x)$ is not 0 for the non-zero values of x for which $J_n'(x) = 0$* .

The proof of the thorem for the case n =,0 in equation (2) follows exactly the same lines. We use, instead of equation (3) the equation

$$\rho\left[J_0'(x)\ N_0'(\rho x) - J_0'(\rho x)\ N_0'(x)\right] = 0. \tag{5}$$

We have thus shown that to each root of $J_n(x) = 0$, (or $J_n'(x) = 0$) there is for small ρ a root of equation (1), [or (2)], which varies continuously with ρ, and which approaches the associated root of J_n (or J_n') as ρ approaches zero.

We now prove that any function $x(\rho)$ arising from some coaxial mode as ρ approaches zero is associated with some root of J_n (or J_n'). As noted in article 4, such an $x(\rho)$ must very continuously with ρ and approach a finite non-zero number as $\rho \to 0$. If $x(\rho)$ is a solution of equation (1) we may consider

$$J_n[x(\rho)] - \frac{J_n[\rho\cdot x(\rho)]}{N_n[\rho\cdot x(\rho)]}\, N_n[x(\rho)] \equiv 0.$$

---

*This can be shown as follows:

Since $J_n(z)$ is a solution of the differential equation

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{n^2}{z^2}\right) u = 0,$$

we have

$$J_n''(z) + \frac{1}{z}J_n'(z) + \left(1 - \frac{n^2}{z^2}\right) J_n(z) = 0.$$

If z_1 is a zero of both J_n' and J_n'', by the preceding relation, $\left(1 - \frac{n^2}{z_1^2}\right) J_n(z_1) = 0$

Then either $J_n(z_1)$ is zero or $1 - \frac{n^2}{z_1^2} = 0$ and $z_1 = \pm n$.

Neither of these possibilities can occur since J_n and J_n' cannot vanish for the same value, and since the smallest positive zero of J_n' is greater than n. A proof of this last statement may be found in Watson's Introduction to Bessel Functions, p. 485.

As ρ approaches 0, x tends to a finite number and so $\dfrac{J_n(\rho x)}{N_n(\rho x)}$ tends to 0 while $N_n(x)$ approaches some finite value. Hence $J_n(x) \rightarrow 0$. Since $J_n(x)$ is continuous, $\lim\limits_{\rho\to 0} x(\rho)$ must be a root of $J_n(x)$. The case of $x(\rho)$ arising from equation (2) is treated in exactly the same way.

Lemma

Let $f(\rho, x)$ and $\dfrac{\partial f(\rho, x)}{\partial x}$ be continuous for all $\rho \geq 0$ and all positive x. It then follows that $\dfrac{\partial f}{\partial x}(0,x) = \lim\limits_{\rho\to 0} \dfrac{\partial f}{\partial x}(\rho, x)$.

Proof:

$$\frac{\partial f}{\partial x}(0,x) = \lim_{\Delta x \to 0} \frac{f(0, x + \Delta x) - f(0,x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \lim_{\rho\to 0} \frac{f(\rho, x + \Delta x) - f(\rho, x)}{\Delta x}.$$

We now apply the mean value theorem to f as a function of x.

$$= \lim_{\Delta x\to 0} \lim_{\rho\to 0} \frac{\partial f}{\partial x}(\rho, x_1) \quad x < x_1 < (x + \Delta x)$$

where as $\Delta x \to 0$, $x_1 \to x$.

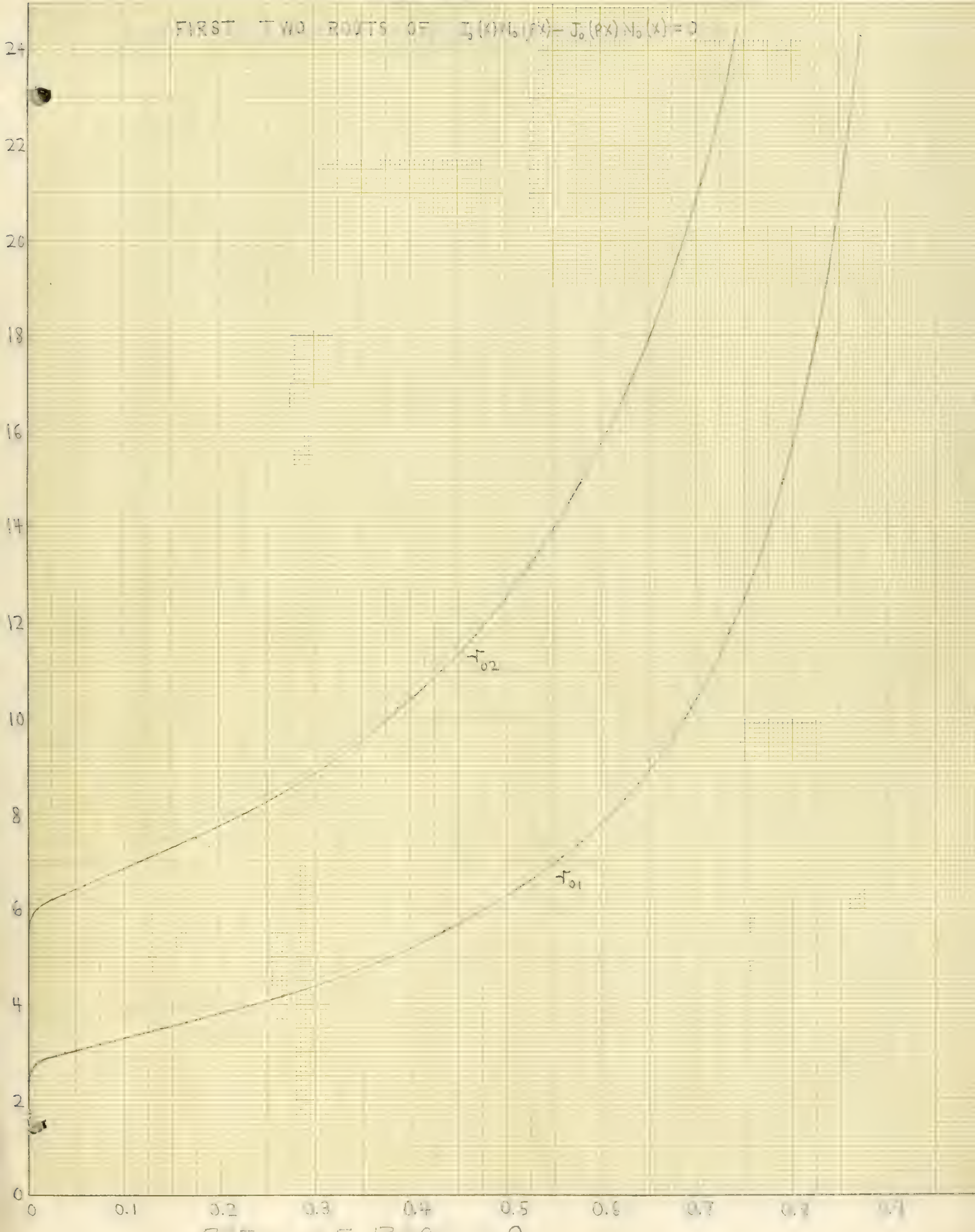
But, because of the continuity of $\dfrac{\partial f}{\partial x}$ at $(0, x)$, this simultaneous limit exists. Hence, we may write

$$\lim_{\Delta x\to 0} \lim_{\rho\to 0} \frac{\partial f}{\partial x}(\rho, x_1) = \lim_{\rho\to 0} \lim_{\Delta x\to 0} \frac{\partial f}{\partial x}(\rho, x_1) = \lim_{\rho\to 0} \frac{\partial f}{\partial x}(\rho, x).$$

This proves the lemma and moreover provides a means of evaluating

$\dfrac{\partial f}{\partial x}(0,x)$.

FIG 2


FIRST TWO ROOTS OF $J_0(x)N_0(\rho x) - J_0(\rho x)N_0(x) = 0$
0
2
4
6
8
10
12
14
16
18
20
22
24
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
x_{02}
x_{01}

— ρ

FIG 2

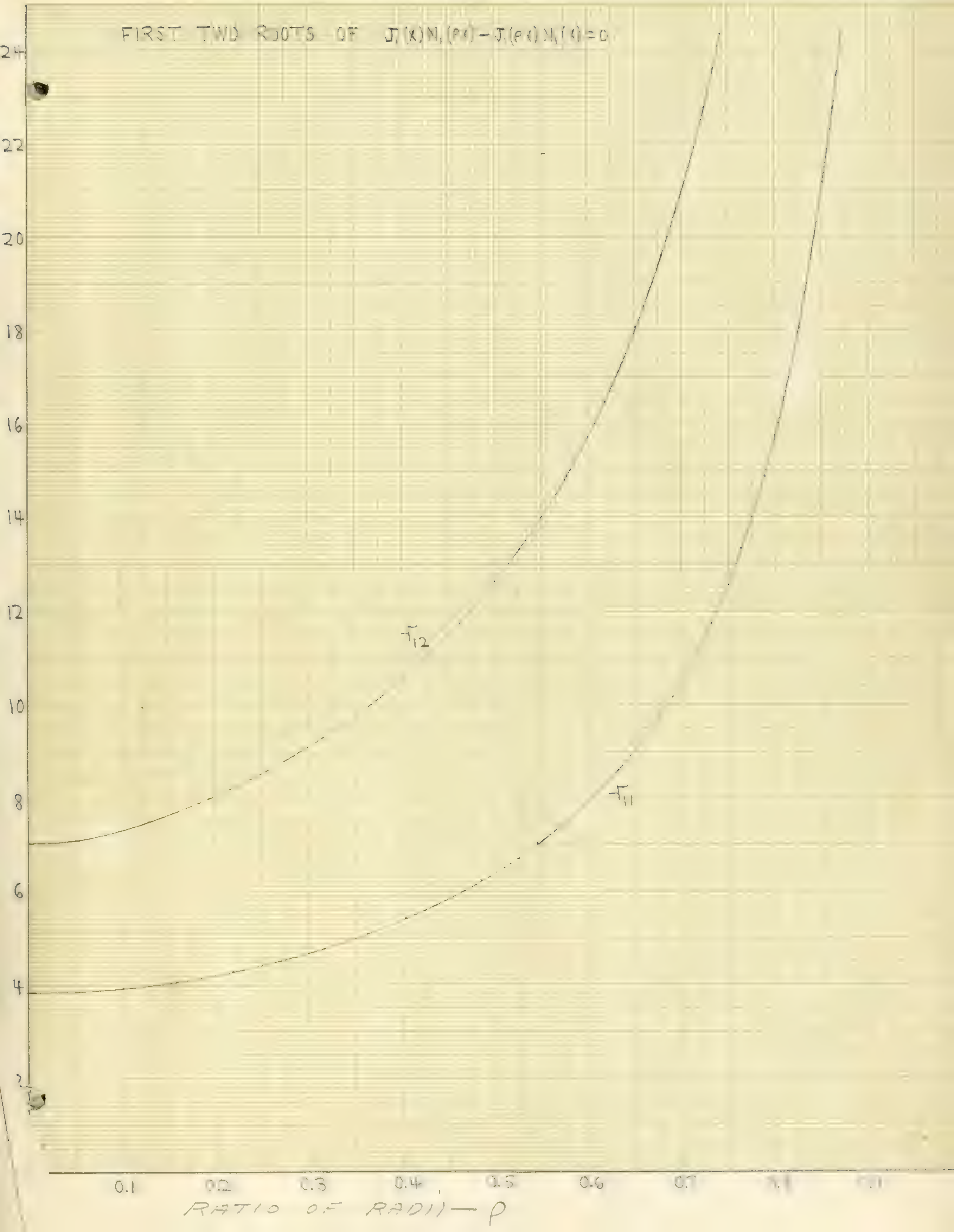
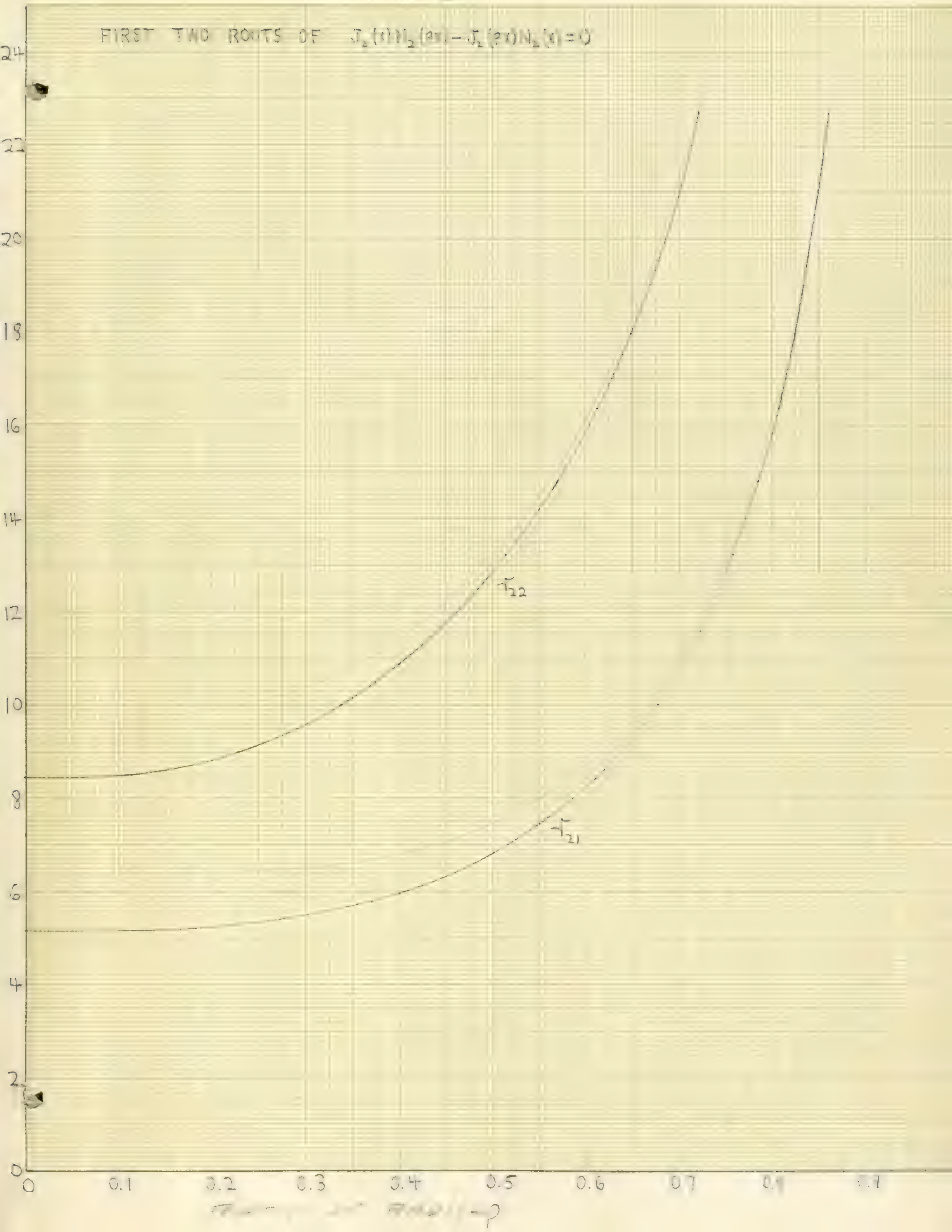
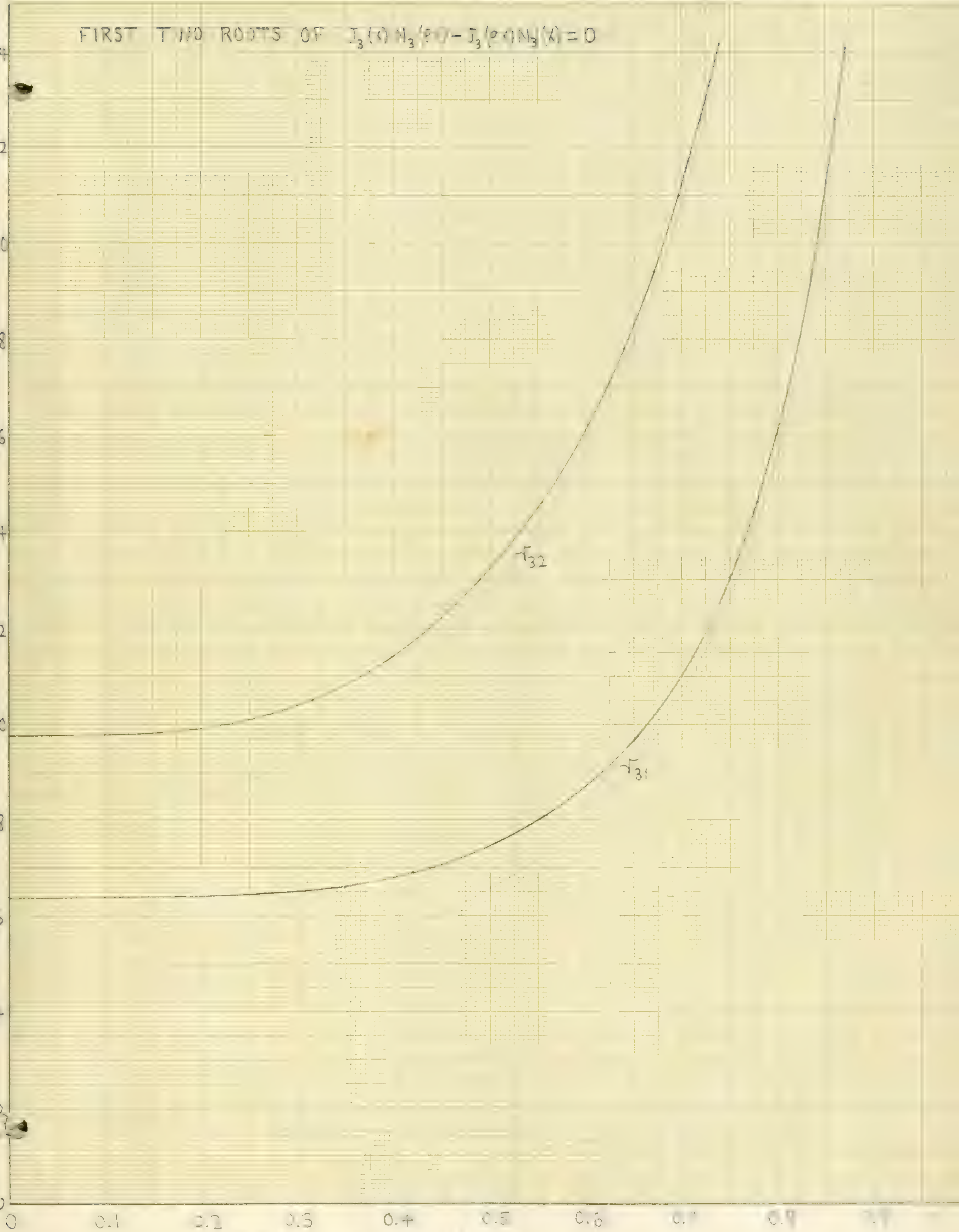

FIRST TWO ROOTS OF $J_1(x)N_1(\rho x) - J_1(\rho x)N_1(x) = 0$
24
22
20
18
16
14
12
10
8
6
4
2
τ_{12}
τ_{11}
0.1
0.2
0.3
0.4
0.5
0.6
0.7
RATIO OF RADII — ρ

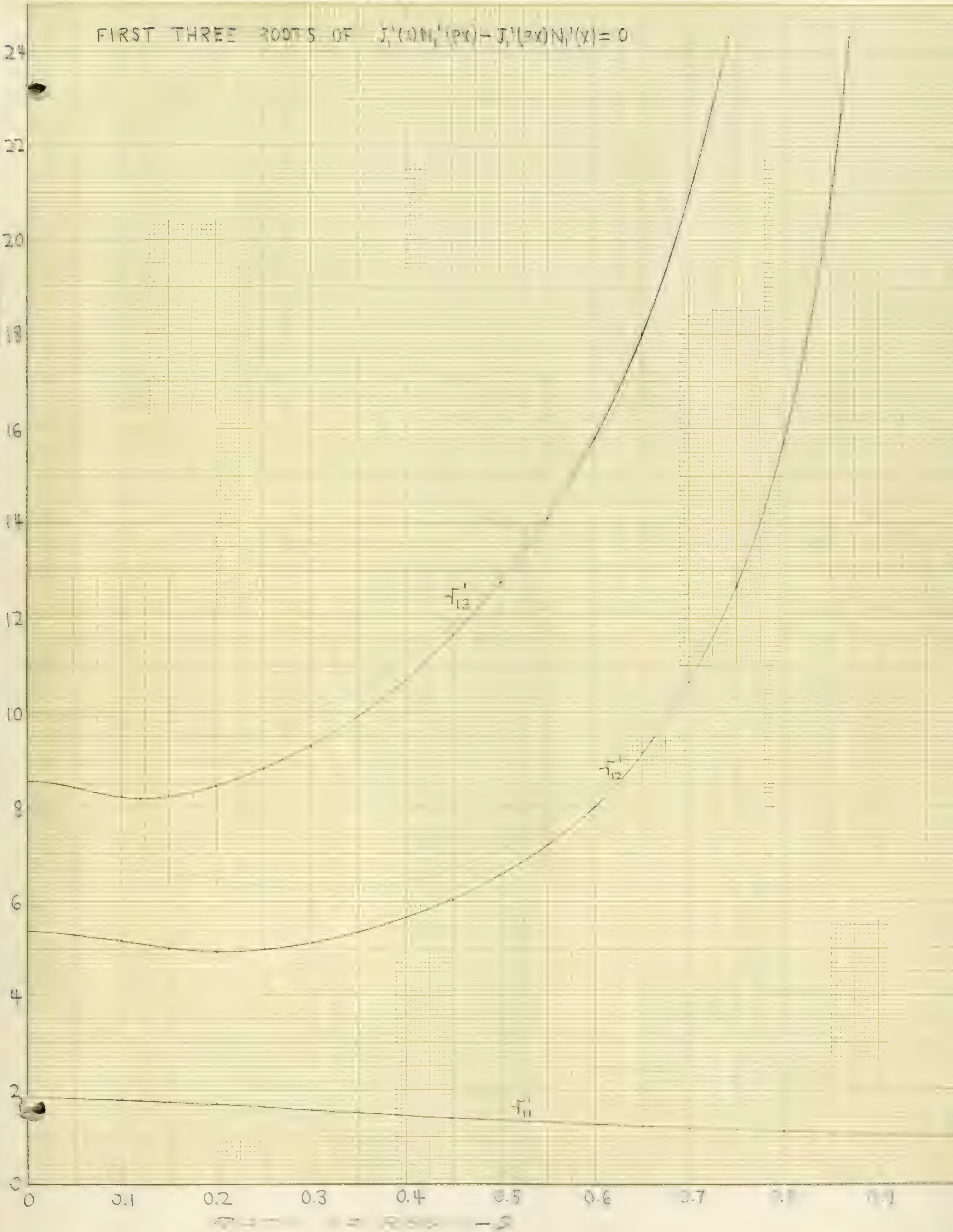
FIG. 3

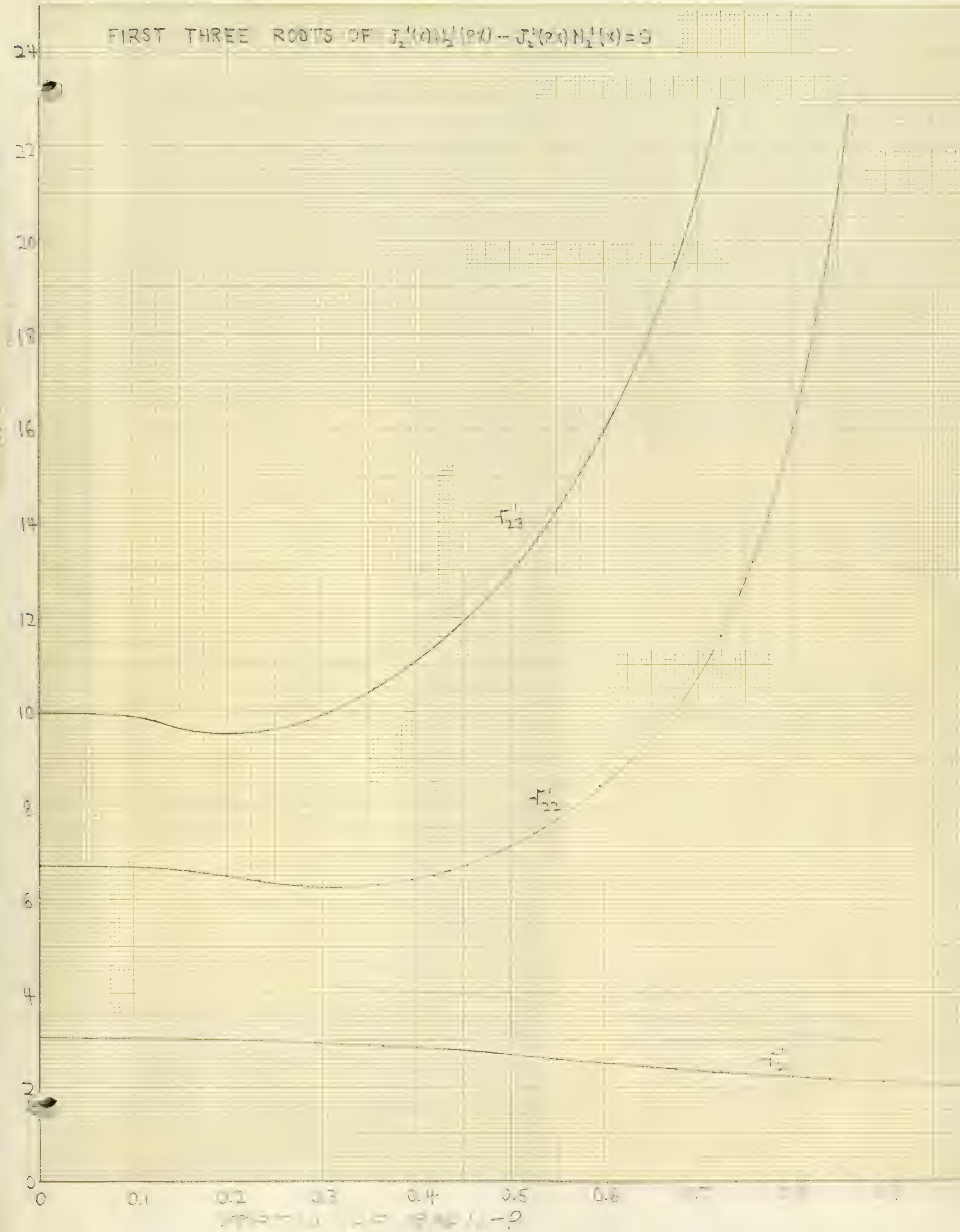


FIRST TWO ROOTS OF $J_2(x)N_2(\rho x) - J_2(\rho x)N_2(x) = 0$
x_{22}
x_{21}
0
2
4
6
8
10
12
14
16
18
20
22
24
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9



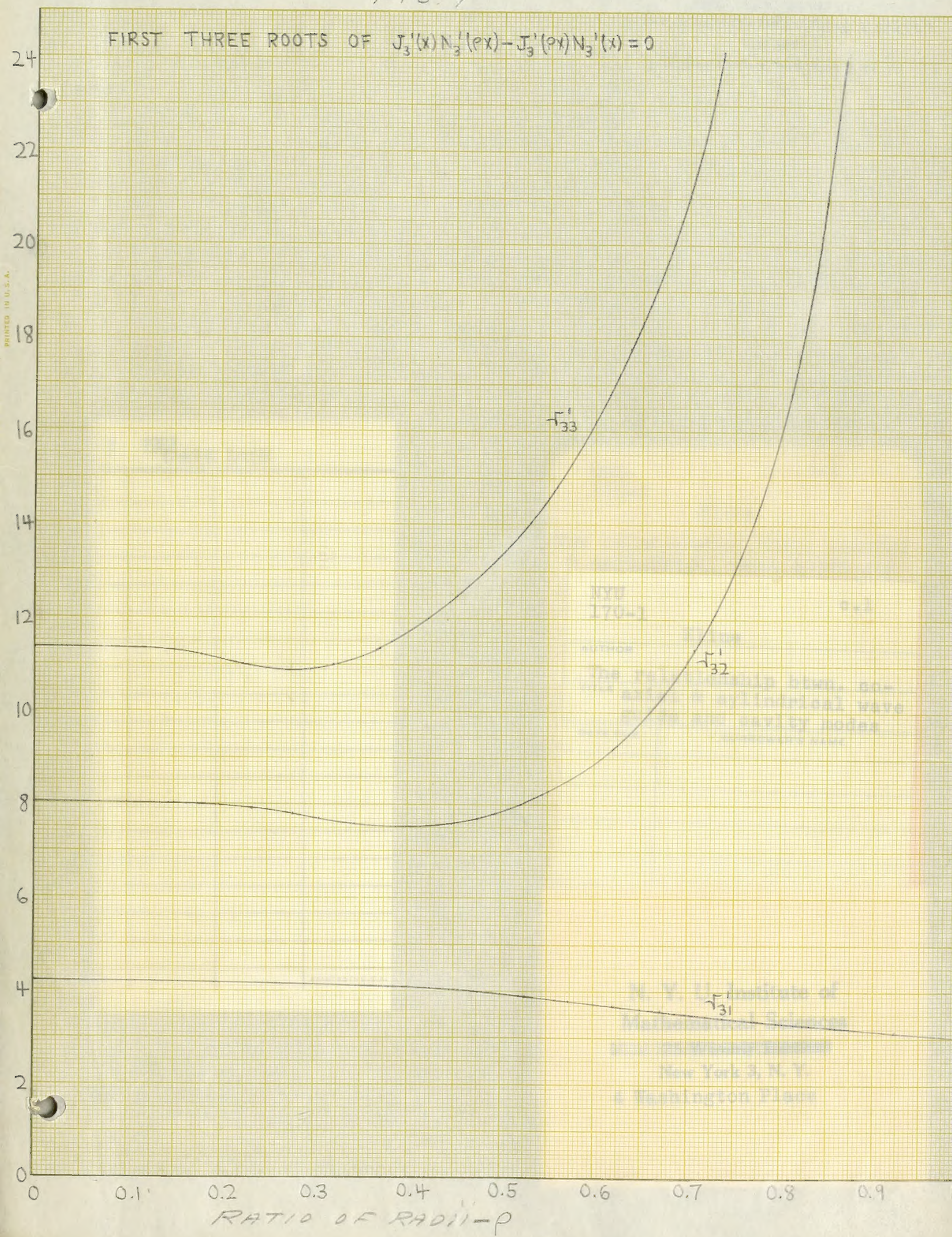
FIRST TWO ROOTS OF $J_3(x)N_3(\rho x) - J_3(\rho x)N_3(x) = 0$
γ_{32}
γ_{31}
0
2
4
6
8
10
12
14
16
18
20
22
24
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9


FIRST THREE ROOTS OF $J_1'(x)N_1'(\rho x) - J_1'(\rho x)N_1'(x) = 0$
24
22
20
18
16
14
12
10
8
6
4
2
0
γ_{13}'
γ_{12}'
γ_{11}'
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9



FIRST THREE ROOTS OF $J_2'(x)N_2'(\rho x) - J_2'(\rho x)N_2'(x) = 0$
24
22
20
18
16
14
12
10
8
6
4
2
0
Γ_{23}'
Γ_{22}'
0
0.1
0.2
0.3
0.4
0.5
0.6

FIG. 7



FIRST THREE ROOTS OF $J_3'(x)N_3'(\rho x) - J_3'(\rho x)N_3'(x) = 0$
Γ_{33}'
Γ_{32}'
Γ_{31}'
24
22
20
18
16
14
12
10
8
6
4
2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
RATIO OF RADII – ρ

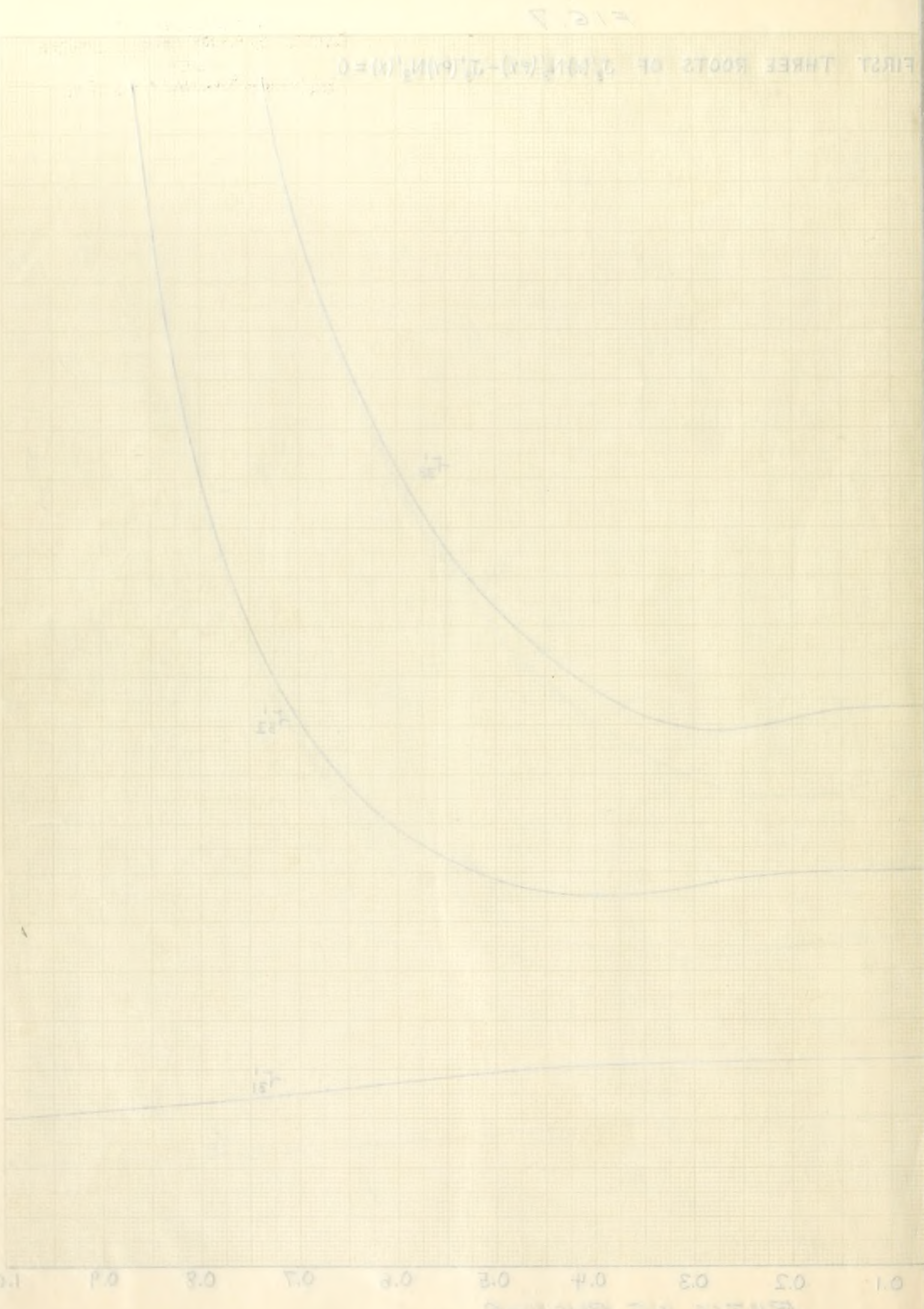


FIG. 7
FIRST THREE ROOTS OF
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

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